

Intrinsic Supersmoothness

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Abstract

The phenomenon, known as “supersmoothness” was first observed for bivariate splines and attributed to the polynomial nature of splines. Using only standard tools from multivariate calculus, we show that if we continuously glue two smooth functions along a curve with a “corner”, the resulting continuous function must be differentiable at the corner, as if to compensate for the singularity of the curve. Moreover, locally, this property, we call supersmoothness, characterizes non-smooth curves. We also generalize this phenomenon to higher order derivatives. In particular, this shows that supersmoothness has little to do with properties of polynomials.

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1 Introduction

In this short article we address supersmoothness: a phenomenon where under certain circumstances continuity of a function of two variables implies its differentiability at a point or, consequently, differentiability of a bivariate function implies its higher order differentiability at a point. Supersmoothness was first observed for a particular class of piecewise bivariate polynomial functions, called splines, by Farin in [2]. He considered a triangle Δ partitioned into three subtriangles Δ_1, Δ_2 and Δ_3 as shown in Figure 1. A spline F on this triangulation of Δ is a function of two variables such that for each $i = 1, 2, 3$, the restriction $F|_{\Delta_i} = f_i$ a polynomial. Farin proved that if the

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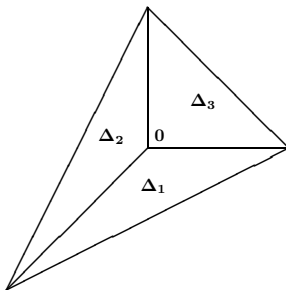


Figure 1: First example

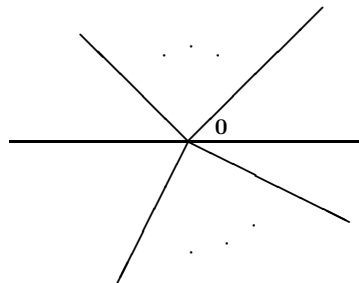


Figure 2: Collinearity matters

spline F is differentiable of order n , then it has all $(n + 1)$ st order partial derivatives at the origin $\mathbf{0} := (0, 0)$. That is for all $n \geq 1$:

$$(1) \quad F \in C^n(\Delta) \Rightarrow F \in C^{n+1}(\mathbf{0}).$$

Supersmoothness of splines was observed for trivariate splines in [1], and studied in general in [3]. This phenomenon has been attributed to the polynomial nature of splines. Recently, while flying on Delta, the authors were struck by the similarity of the emblem of the airline and Figure 1. This lead us to the subject of this paper.

In the next section we will demonstrate that basic supersmoothness is a rather general property of non-smooth curves, not just polynomials. Loosely speaking: if we want to continuously glue two smooth bivariate functions along a curve with a “corner” at a point P , the resulting continuous function must be differentiable at P , as if to compensate for the singularity of the curve. Moreover, locally, supersmoothness characterizes non-smooth curves.

In Section 3 we address another peculiarity of supersmoothness. We first show that property (1) holds for all smooth functions defined over a partition of \mathbb{R}^2 by $n + 2$ non-collinear rays emanating from the origin, $n \geq 0$. The assumption that the rays are not collinear is significant. If just two of the rays are parallel the phenomenon of automatic supersmoothness disappears altogether. This can be seen on the following simple example. Consider the partition of \mathbb{R}^2 by the x -axis. For any $n \geq 0$, let $f(x, y)$ be equal to y^{n+1} on the upper half plane and zero on the lower one. We now can add any n rays emanating from the origin but not along the x -axis. This will form a

partition of \mathbb{R}^2 by $n + 2$ rays as in Figure 2. Then f has all derivatives of order n , yet $f \notin C^{n+1}(\mathbf{0})$.

We note that all the proofs in this article are simple and should be accessible to undergraduate students familiar with the basics of multivariate calculus, such as in [4].

2 Gluing functions along a curve

In this section we will show that a version of supersmoothness occurs when we glue two differentiable functions along a curve with sharp corners as in Figure 3. Namely, we will show that the resulting piecewise function is differentiable at every sharp corner of the curve. To some extent this property of supersmoothness characterizes curves with sharp corners.

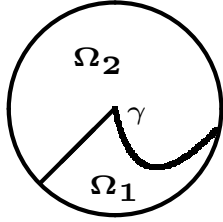


Figure 3: Curves for gluing

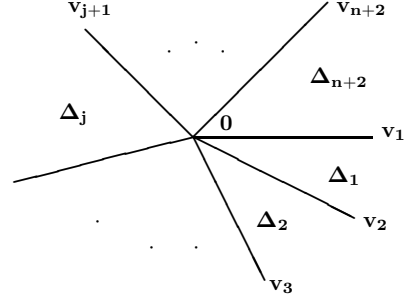


Figure 4: Supersmoothness of higher derivatives

In contrast with most of the research on curves in both analysis and differential geometry, we are interested in non-smooth curves. While regularity of a curve is defined globally, non-smoothness has to be localized to a point $P := (x_0, y_0)$. Recall (cf. [4]) that a curve

$$\gamma(t) := (u(t), v(t)) : [a, b] \rightarrow \mathbb{R}^2$$

is regular if γ is differentiable and $\gamma'(t) \neq 0$ for all t . In particular, for every point t at least one of the derivatives, say $u'(t) \neq 0$. Hence the function $u(t)$ is invertible in a neighborhood of that point. Setting $u(t) = x$ we have

$t = u^{-1}(x)$ and we can reparametrize a portion of the curve as $(x, f(x))$ where $f := v \circ u^{-1}$.

To keep this article within reach of calculus students we limit our considerations to non-self-intersecting curves and adopt the following, intuitively clear version of “local smoothness”. Let γ be the trace of a continuous non-self-intersecting curve $\gamma(t) : [a, b] \rightarrow \mathbb{R}^2$, also known as a Jordan arc. Without loss of generality assume that $\gamma(0) = P$ and $a < 0 < b$.

We shall say that γ is smooth at P if γ can be represented as a graph of a continuously differentiable function in some neighborhood of P . More precisely,

Definition 1 *The trace of a Jordan arc γ is smooth at a point P if there exist open intervals I, J and a function $f \in C^1(I)$ such that*

$$P = (x_0, f(x_0)) \in I \times J, \quad \text{and} \quad \gamma \cap (I \times J) = \{(x, f(x)), \quad x \in I\}.$$

Theorem 2 *The trace of a Jordan arc γ is smooth at P if and only if there exists a neighborhood U of P and a function h continuously differentiable on U such that*

$$(2) \quad h(x, y) = 0 \quad \text{if} \quad (x, y) \in \gamma \cap U, \quad \text{and} \quad \nabla h(P) \neq \mathbf{0}.$$

Proof. If γ is smooth at P we use the neighborhood $I \times J$ and the C^1 -continuous function f from Definition 1 to construct $h(x, y) := y - f(x)$. Clearly, h satisfies all the desirable properties. Conversely, without loss of generality, assume $P = (0, 0)$ and let h be a C^1 -continuous function on some neighborhood U of P , such that h vanishes on γ , and $h_y(P) \neq 0$. Then by the Implicit Function Theorem, there exist open intervals I_1 and J_1 and a C^1 -continuous function f such that

$$h(x, y) = 0, \quad (x, y) \in I_1 \times J_1 \quad \text{iff} \quad y = f(x), \quad x \in I_1, \quad y \in J_1.$$

We can assume that $I_1 \times J_1 \subset U$, which implies that

$$\gamma \cap (I_1 \times J_1) \subseteq \{(x, f(x)), \quad x \in I_1\}.$$

We now need to show that there exist perhaps other intervals $I \subseteq I_1$ and $J \subseteq J_1$ such that γ coincides with the graph of f in $I \times J$. To this end, we consider the inverse image $\gamma^{-1}(I_1 \times J_1)$, which is an open set in $[a, b]$ containing zero. Thus, there exists $c > 0$ such that $\gamma(t) := (u(t), v(t))$ maps $(-c, c)$ into $\gamma \cap (I_1 \times J_1)$. We observe that if $u(c/2) = 0$, then $v(c/2)$ also vanishes since f is a function passing through $(0, 0)$. Then we have $\gamma(c/2) =$

$\gamma(0)$ which contradicts the assumption that γ has no self-intersections. Thus, neither $u(c/2)$ nor $u(-c/2)$ vanish. Without loss of generality we can assume $u(c/2) > 0$. Then $u(-c/2)$ must be negative. Otherwise either γ has a self-intersection or f is not a function. Since $\gamma(t)$ is continuous, its trace from $t = -c/2$ to $t = c/2$ must coincide with the graph of f from $u(-c/2) < 0$ to $u(c/2) > 0$. Thus, for $I := (u(-c/2), u(c/2))$, and $J := J_1$, the function f satisfies Definition 1. ■

As a corollary we obtain the promised result on supersmoothness:

Theorem 3 *Let $\gamma \subset \mathbb{R}^2$ be the trace of a Jordan arc that divides the open disk Ω into two subsets Ω_1 and Ω_2 as in Figure 3. Further assume that γ is not smooth at $P \in \gamma$. Let f_1, f_2 be C^1 functions on Ω continuously glued along γ , that is, let*

$$(3) \quad F(x, y) := \begin{cases} f_1(x, y) & \text{if } (x, y) \in \Omega_1 \\ f_2(x, y) & \text{if } (x, y) \in \Omega_2 \end{cases}$$

be a continuous function on Ω . Then the piecewise function F is differentiable at P , that is,

$$(4) \quad \nabla f_1(P) = \nabla f_2(P).$$

Proof. Consider a C^1 function $h = f_1 - f_2$. The fact that f_1 and f_2 are continuously glued along γ means that $h(\gamma) = 0$ and by Theorem 2

$$0 = \nabla h(P) = \nabla f_1(P) - \nabla f_2(P).$$

Thus, $\nabla f_1(P) = \nabla f_2(P)$, and the proof is complete. ■

A partial converse of Theorem 3 holds true in the following sense:

Theorem 4 *Let $\gamma \subset \mathbb{R}^2$ be the trace of a Jordan arc that divides the open disk Ω into two subsets Ω_1 and Ω_2 . Assume that γ is smooth at a point $P \in \gamma$. Then there exists a neighborhood U of P and two differentiable functions $f_1, g_1 \in C^1(U)$ such that the function*

$$(5) \quad F(x, y) := \begin{cases} f_1(x, y) & \text{if } (x, y) \in \Omega_1 \\ f_2(x, y) & \text{if } (x, y) \in \Omega_2 \end{cases}$$

is not differentiable at P .

Proof. Let U and h be chosen as in Theorem 2, i.e., satisfying conditions (2). Let $f_1(x, y) := h(x, y)$ and $f_2(x, y) \equiv 0$. Then, since $h(\gamma \cap U) = 0$, the

function F defined by (5) is continuous and not differentiable at P because $\nabla f_2(P) = \mathbf{0} \neq \nabla f_1(P)$. ■

Theorem 4 provides only a partial converse of Theorem 3 because the function F is defined locally, in a neighborhood U of P , and not on all of Ω . We believe that the global version of this theorem also holds and end this section with a conjecture.

Conjecture 5 *Let $\gamma \subset \mathbb{R}^2$ be a continuous curve that divides an open disk Ω centered at P into two subsets Ω_1 and Ω_2 . Then γ is smooth at P if and only if we can glue two continuously differentiable functions along the curve as in (5) so that the resulting piecewise function F is not differentiable at P .*

3 Supersmoothness of higher derivatives.

Consider two non-collinear rays v_1 and v_2 emanating from the origin in \mathbb{R}^2 . The curve formed by these two rays is not smooth and partitions the open unit disk Ω into two sectors Δ_1 and Δ_2 . It follows from the results of the previous section that two differentiable functions f_1 and f_2 continuously glued along the boundary of the sectors as in (5) produce a piecewise function F_2 differentiable at the origin:

$$(6) \quad F_2 \in C(\Omega) \Rightarrow F_2 \in C^1(\mathbf{0}).$$

Farin's observation (1) shows that for three pairwise non-collinear rays emanating from the origin and a piecewise function F_3 consisting of three differentiable pieces as in Figure 1 the following holds:

$$F_3 \in C^1(\Omega) \Rightarrow F_3 \in C^2(\mathbf{0}).$$

However, as it was pointed out in the introduction, for three non-collinear rays amplification (6) may not hold, that is, in general

$$F_3 \in C(\Omega) \not\Rightarrow F_3 \in C^1(\mathbf{0}).$$

In this section we extend this pattern. For a fixed $n \geq 0$, we partition the open disk Ω into $n+2$ sectors $\Delta_1, \dots, \Delta_{n+2}$, by pairwise non-collinear vectors (rays) v_1, \dots, v_{n+2} , positioned clockwise as in Figure 4. Then we create a piecewise function F_{n+2} by gluing $n+2$ functions $f_1, \dots, f_{n+2} \in C^n(\Omega)$ along the rays. Thus for $1 \leq j \leq n+1$, the sector Δ_j is formed by v_j and v_{j+1} , and the sector Δ_{n+2} is formed by v_{n+2} and v_1 . We will show that similarly to (1) the following holds:

$$(7) \quad F_{n+2} \in C^n(\Omega) \Rightarrow F_{n+2} \in C^{n+1}(\mathbf{0});$$

yet the weaker assumption $F_{n+2} \in C^{n-1}(\Omega)$ may not imply the associated conclusion that $F_{n+2} \in C^n(\mathbf{0})$.

We start with a simple lemma that shows that two differentiable functions continuously glued along a ray v must be differentiable in the direction of v . We use D_v to denote the directional derivative in the direction of v .

Lemma 6 *Let $v = (a, b)$ be a unit vector in \mathbb{R}^2 . Let f and g be continuously differentiable functions in an ε -neighborhood of the origin in \mathbb{R}^2 such that*

$$(8) \quad f(ta, tb) = g(ta, tb), \quad \text{for all } t \in [0, \varepsilon).$$

Then

$$D_v f(ta, tb) = D_v g(ta, tb), \quad \text{for all } t \in [0, \varepsilon).$$

Proof. It suffices to prove the result for $t = 0$. We obtain

$$\begin{aligned} D_v f(\mathbf{0}) &= \lim_{t \rightarrow 0} \frac{f(ta, tb) - f(\mathbf{0})}{t} = \lim_{t \rightarrow 0+} \frac{f(ta, tb) - f(\mathbf{0})}{t} \\ &\stackrel{\text{by (8)}}{=} \lim_{t \rightarrow 0+} \frac{g(ta, tb) - g(\mathbf{0})}{t} = \lim_{t \rightarrow 0} \frac{g(ta, tb) - g(\mathbf{0})}{t} = D_v g(\mathbf{0}), \end{aligned}$$

where the second and the fourth equalities follow from the continuity of $D_v f$ and $D_v g$, respectively. ■

We are now ready to prove statement (7). For brevity, we use $F := F_{n+2}$.

Theorem 7 *Let functions f_1, \dots, f_{n+2} , be n times continuously differentiable on Ω and let F be defined piecewise on each sector Δ_j by $F|_{\Delta_j} := f_j$, $j = 1, \dots, n+2$. If $F \in C^n(\Omega)$ then F has all derivatives of order $n+1$ at the origin; that is, $F \in C^{n+1}(\mathbf{0})$, $n \geq 0$.*

Proof. If $n = 0$, the proof is given in Theorem 3. Let $n \geq 1$. We will show that for two neighboring functions, say f_j and f_{j+1} , all partial derivatives of order $n+1$ coincide at the origin. Then for every $k = 0, \dots, n$,

$$D_x^k D_y^{n-k} f_1(\mathbf{0}) = D_x^k D_y^{n-k} f_2(\mathbf{0}) = \dots = D_x^k D_y^{n-k} f_{n+2}(\mathbf{0}),$$

which would prove the theorem. Without loss of generality we consider the neighboring functions f_1 and f_2 . It is clearly enough to prove that

$$(9) \quad D_{v_2}^k D_{v_1}^{n-k} f_1(\mathbf{0}) = D_{v_2}^k D_{v_1}^{n-k} f_2(\mathbf{0}), \quad \text{for every } k = 0, \dots, n.$$

Observe that for $k \geq 1$, the assumption $F \in C^n(\Omega)$ implies that the functions $D_{v_2}^{k-1} D_{v_1}^{n-k} f_1$ and $D_{v_2}^{k-1} D_{v_1}^{n-k} f_2$ are continuously glued along the ray v_2 . Hence, by Lemma 6 we obtain

$$D_{v_2}(D_{v_2}^{k-1} D_{v_1}^{n-k} f_1)(\mathbf{0}) = D_{v_2}(D_{v_2}^{k-1} D_{v_1}^{n-k} f_2)(\mathbf{0})$$

which implies (9) for $k \geq 1$. Hence it remains to prove that

$$(10) \quad D_{v_1}^n f_1(\mathbf{0}) = D_{v_1}^n f_2(\mathbf{0}).$$

Since all the vectors v_j are pairwise non-collinear we can find constants α_j and β_j such that $v_1 = \alpha_j v_2 + \beta_j v_j$ for all $j = 3, \dots, n+2$. Then

$$(11) \quad \begin{aligned} D_{v_1}^n &= (\alpha_3 D_{v_2} + \beta_3 D_{v_3}) \dots (\alpha_{n+2} D_{v_2} + \beta_{n+2} D_{v_{n+2}}) \\ &= D_{v_2} p(D_{v_2}, \dots, D_{v_{n+2}}) + \gamma \prod_{j=3}^{n+2} D_{v_j} \end{aligned}$$

for some constant γ and some homogeneous polynomial p of order $n-1$. Since, by the assumption, $p(D_{v_2}, \dots, D_{v_{n+2}}) f_1$ and $p(D_{v_2}, \dots, D_{v_{n+2}}) f_2$ coincide along the ray v_2 , by Lemma 6

$$(12) \quad D_{v_2} p(D_{v_2}, \dots, D_{v_{n+2}}) f_1(\mathbf{0}) = D_{v_2} p(D_{v_2}, \dots, D_{v_{n+2}}) f_2(\mathbf{0}).$$

Similarly, for every $k = 3, \dots, n+2$, the functions

$$\prod_{j=3, j \neq k}^{n+2} D_{v_j} f_{k-1} \quad \text{and} \quad \prod_{j=3, j \neq k}^{n+2} D_{v_j} f_k$$

coincide along the ray v_k . Hence, by Lemma 6, for every $k = 3, \dots, n+2$,

$$\prod_{j=3}^{n+2} D_{v_j} f_{k-1}(\mathbf{0}) = D_{v_k} \prod_{\substack{j=3 \\ j \neq k}}^{n+2} D_{v_j} f_{k-1}(\mathbf{0}) = D_{v_k} \prod_{\substack{j=3 \\ j \neq k}}^{n+2} D_{v_j} f_k(\mathbf{0}) = \prod_{j=3}^{n+2} D_{v_j} f_k(\mathbf{0}).$$

Thus, we obtain the following chain of equalities

$$(13) \quad \prod_{j=3}^{n+2} D_{v_j} f_2(\mathbf{0}) = \prod_{j=3}^{n+2} D_{v_j} f_3(\mathbf{0}) = \dots = \prod_{j=3}^{n+2} D_{v_j} f_{n+2}(\mathbf{0}) = \prod_{j=3}^{n+2} D_{v_j} f_1(\mathbf{0}).$$

The last equality follows from Lemma 6 since f_1 and f_{n+2} share a common edge v_1 . Thus

$$\begin{aligned} D_{v_1}^n f_2(\mathbf{0}) &\stackrel{\text{by (11)}}{=} D_{v_2} p(D_{v_2}, \dots, D_{v_{n+2}}) f_2(\mathbf{0}) + \gamma \prod_{j=3}^{n+2} D_{v_j} f_2(\mathbf{0}) \\ &\stackrel{\text{by (12,13)}}{=} D_{v_2} p(D_{v_2}, \dots, D_{v_{n+2}}) f_1(\mathbf{0}) + \gamma \prod_{j=3}^{n+2} D_{v_j} f_1(\mathbf{0}) \stackrel{\text{by (11)}}{=} D_{v_1}^n f_1(\mathbf{0}). \end{aligned}$$

which completes the proof of (9), and consequently proves the theorem. ■

The next result is a direct consequence of applying Theorem 7 to the derivatives of the piecewise function.

Corollary 8 *Let functions f_1, \dots, f_{n+2} , be m times continuously differentiable on Ω , with $m \geq n$, and let F_{n+2} be defined piecewise on each sector Δ_j by $F_{n+2}|_{\Delta_j} = f_j$, $j = 1, \dots, n+2$. If $F_{n+2} \in C^m(\Omega)$ then F_{n+2} has all derivatives of order $m+1$ at the origin, that is, $F_{n+2} \in C^{m+1}(\mathbf{0})$, $m \geq n \geq 0$.*

We finish this section and this article by constructing polynomials (hence smooth functions) f_1, \dots, f_{n+2} , $n \geq 1$, such that the spline F_{n+2} defined by $F_{n+2}|_{\Delta_j} = f_j$ is in $C^{n-1}(\Omega)$ yet $F_{n+2} \notin C^n(\mathbf{0})$. We note that if $n = 0$, it is immediately obvious that $f_1 \equiv 0$ and $f_2 \equiv 1$ do not join continuously at the origin. The following observation is the key to the construction:

Lemma 9 *Given $n \geq 1$, consider the polynomial*

$$g(x, y) := \sum_{i=1}^{n+1} c_i (y + a_i x)^n.$$

Then the system of equations with the unknowns (c_1, \dots, c_{n+1}) :

$$\frac{\partial^k}{\partial x^j \partial y^{k-j}} g(x, 0) = 0, \quad \text{for all } 0 \leq j \leq k \leq n-1,$$

has a non-trivial solution.

Proof. Indeed for $0 \leq k \leq n-1$ and $0 \leq j \leq k$ we have

$$\begin{aligned} \sum_{i=1}^{n+1} c_i \frac{\partial^k}{\partial x^j \partial y^{k-j}} (y + a_i x)^n \Big|_{y=0} &= \frac{n!}{(n-k)!} \sum_{i=1}^{n+1} c_i a_i^j (y + a_i x)^{n-k} \Big|_{y=0} \\ &= \frac{n!}{(n-k)!} x^{n-k} \sum_{i=1}^{n+1} c_i a_i^{n-k+j} = 0. \end{aligned}$$

With $s := n - k + j$, the system of n equations with $n + 1$ unknowns

$$\sum_{i=1}^{n+1} c_i a_i^s = 0, \quad s = 0, \dots, n-1,$$

has a non-trivial solution. ■

Now we can proceed with our construction. As in Figure 4, choose $n + 2$ consecutive positioned clockwise rays v_i emanating from the origin whose equations are given by the following lines l_i

$$l_1 : y = 0, \quad l_2 : y + a_2 x = 0, \quad \dots, \quad l_{n+2} : y + a_{n+2} x = 0.$$

Note that without loss of generality we assume that v_1 goes along the positive direction of the x -axis. Define $f_1 \equiv 0$ to be the function between v_1 and v_2 . Let the function between v_k and v_{k+1} be defined as follows:

$$f_k := \sum_{i=2}^k c_i l_i^n, \quad \text{for each } 2 \leq k \leq n+2,$$

with the convention $v_{n+3} := v_1$. We next define:

$$g_k(x, y) := f_{k+1}(x, y) - f_k(x, y) = c_{k+1}(y + a_{k+1}x)^n, \quad \text{for all } 2 \leq k \leq n+1.$$

All partial derivatives of g_k of order $n - 1$ or less vanish for $y = -a_{k+1}x$, that is, at the line l_{k+1} . It remains to choose the coefficients c_2, \dots, c_{n+2} in such a way that f_{n+2} is glued smoothly to $f_1 \equiv 0$ at l_1 , that is, so that all derivatives of order $n - 1$ or less of the polynomial

$$f_{n+2} = \sum_{i=2}^{n+2} c_i l_i^n$$

vanish at $y = 0$. By Lemma 9 this leads to a system of n equations with $n + 1$ unknowns (c_2, \dots, c_{n+2}) that has a nontrivial solution.

Hence there exists a non-zero homogeneous polynomial f_{n+2} of order n between l_{n+2} and l_1 which is C^{n-1} -smoothly glued to f_{n+1} across l_{n+1} and C^{n-1} -smoothly glued to $f_1 \equiv 0$ across l_1 . Finally f_{n+2} is a nonzero homogeneous polynomial of order n . Thus there exists a partial derivative of f_{n+2} of order n which is a non-zero constant. In particular, its value at the origin is not zero, yet the same derivative of $f_1 \equiv 0$ is zero. The resulting piecewise function F_{n+2} does not have a derivative of order n at the origin.

Remark 10 *The existence of the spline F_{n+2} implicitly constructed above also follows from Theorem 9.3 in [5]. Indeed this theorem shows that the dimension of polynomial splines of degree n and smoothness $n - 1$ defined over the union of $n + 2$ sectors is strictly greater than $\binom{n+2}{2}$. The latter is the dimension of bivariate polynomials. Thus, there exists a spline that does not have a derivative of order n at the origin. We decided to provide a development here that would be accessible to an audience not familiar with spline theory.*

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